Uva Wellassa University of Sri Lanka
Faculty of Science and Technology
Department of Science and Technology
100 Level 2nd Semester Examination – Dec./Jan. 2016/17
SCT 102-3 Calculus

Uva Wellassa University

**Instructions to candidates**
**Duration:** Three(03) hours
**Number of questions:** Six(06) Essay Questions
**Mark allocation:** 200 mark
Use standard symbols without definition.
Scientific calculators are allowed.
**Answer all questions.**

1.
a. Evaluate the following limits.

i. $\lim_{x\to 3}\left(x^2-2x+3\right)$ (02 mark)

ii. $\lim_{x\to -3}\frac{x^2+10x+21}{x+3}$ (03 mark)

iii. $\lim_{x\to 0}\frac{\sqrt{2x+1}-1}{x}$ (04 mark)

iv. $\lim_{x\to\infty}\frac{2x^{2017}+3x^{2015}-6x^{1988}+3}{x^{2017}-5x^{2014}-x^{1992}-4}$ (04 mark)

v. $\lim_{x\to -2}\frac{x^3+8}{x+2}$ (04 mark)

b. Show that $\lim_{x\to 0}\frac{\tan^2 3x}{x\left(\sqrt{1+x}-1\right)}=18.$ (04 mark)

c. State the **Sandwich theorem** and prove $\lim_{x\to\infty}\frac{\sin x}{x}=0.$ (05 mark)

d. Let $f(x)=\begin{cases}1-3x, & x<0\\ 1, & x=0\\ 1+5x, & x>0\end{cases}$, then show that $\lim_{x\to 0} f(x)$ exists and is equal to 1. (04 mark)

e. Show that the function $f$ defined by,

$$f(x)=\begin{cases}\left(x^2/a\right)-a & ; \ 0<x<a\\ 0 & ; \ x=a\\ a-\left(a^3/x^2\right) & ; \ x>a\end{cases}$$

is continuous at $x=a$. (05 mark)





10

2.

a. Use the *definition of derivative* to prove $\frac{d}{dx}(\sin x) = \cos x$. (03 mark)

b. Find $\frac{dy}{dx}$, for each of the following $y$.

i. $y = 4x^5 + 2x^3 - 5x + 7$ (03 mark)

ii. $y = (x^2 + 6x - 5)^{18}$ (03 mark)

iii. $y = (3x^2 + 2)(5x + 7)$ (04 mark)

iv. $y = \frac{3x^3 - 2}{x^2 + 1}$ (04 mark)

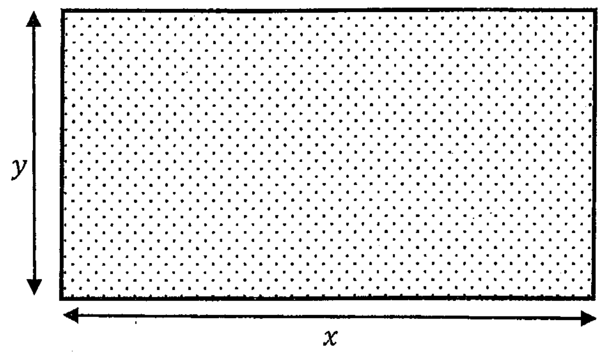
v. $y = \ln(\sin^2(2x^3 + 4))$ (04 mark)

c. Let $f(x) = 2x^3 + ax^2 + bx - 8$ for $x \in \mathbb{R}$, where $a$ and $b$ are real constants. Suppose that $f'(1) = 16$ and $f''(1) = 18$, where $f'$ and $f''$ have usual meaning. Find the values of $\boldsymbol{a}$ and $\boldsymbol{b}$. (05 mark)

d. Find $\frac{dy}{dx}$, given that $x^2 + 3xy - y^2 + 2x - 3y + 7 = 0$. (04 mark)

e. Sketch the graph of $y = \frac{1}{3}x^3 - 2x^2 + 3x + 1$ for $-\infty < x < +\infty$, using **first derivative test** or **second derivative test.** (10 mark)

f. A rectangular garden along of one side of a house is shown in the following diagram. The dimensions in meters of the garden are indicated there. The area of the garden is 216 m$^2$.




i. Express $y$ in terms of $x$. (02 mark)

ii. Show that the perimeter ($P$) of the garden (measured in meters) , is given by $P = 2x + \frac{432}{x}$. (03 mark)

iii. Hence, find the minimum value of the perimeter of the garden. (05 mark)

3.

a. Integrate the following functions with respect to $x$.

i. $\int\left(x^8+6x^5-3x^2+5\right)dx$ (02 mark)

ii. $\int(3x+7)^5\,dx$ (02 mark)

iii. $\int\sin(5x-3)\,dx$ (02 mark)

iv. $\int e^{x+3}\,dx$ (02 mark)

v. $\int\frac{1}{\sqrt{4-x^2}}\,dx$ (02 mark)

b. Find constants $A$ and $B$ such that $4e^x+5e^{-x}=A(4e^x-e^{-x})+B(4e^x+e^{-x})$.

**Hence,** evaluate $\int\frac{4e^x+5e^{-x}}{4e^x+e^{-x}}\,dx.$ (04 mark)

c. Find the values of constants $A$, $B$ and $C$ such that $\frac{x^2-1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$.

**Hence,** evaluate $\int\frac{x^2-1}{x\left(x^2+1\right)}\,dx$. (05 mark)

d. Using the **substitution** $u=3x+1$, show that $\int\frac{x}{(3x+1)^2}\,dx=\frac{1}{9}\left\{\ln(3x+1)+\frac{1}{3x+1}\right\}+C$, where $C$ is an arbitrary constant. (04 mark)

e. Compute the integral;

$$I=\int_0^1 x\tan^{-1}x\,dx$$, using **integration by parts.** (04 mark)

f. Find the area of the region ($A$) in the figure 01, that is bounded by the curve $y=2x^2$ and $y=3x+2$. (03 mark)

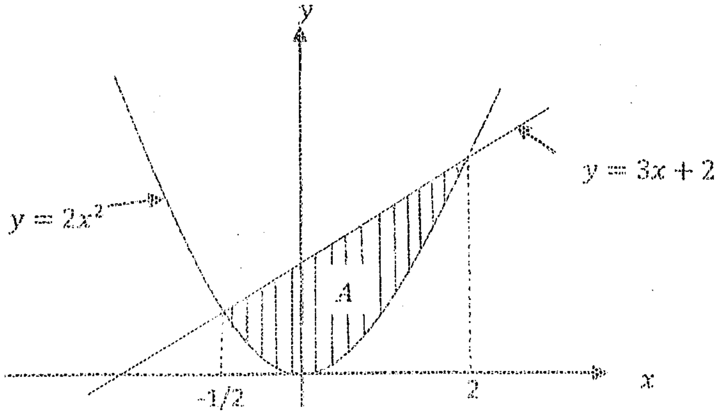



Figure 01: Graphs of $y = x^2$ and $y = 3x + 2$

4.

a. Obtain the first four terms of the following sequences.

i. $\left\{\frac{n}{n+1}\right\}, n \in \mathrm{N}$ (02 mark)

ii. $\left\{\frac{(-1)^n}{n}\right\}, n \in \mathrm{N}$ (02 mark)

b. Determine whether the following sequences are *convergent* or *divergent*. **Justify your answer.**

i. $\left\{\frac{n^2+3n+5}{2n^2+5n+7}\right\}, n \in \mathrm{N}$ (03 mark)

ii. $\left\{1+(-1)^n\right\}, n \in \mathrm{N}$ (03 mark)

c. Determine whether each of the following series is *convergent* or *divergent*. **Justify your answer.**

i. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots$ (03 mark)

ii. $1+3+9+27+\ldots\ldots\ldots$ (02 mark)

d. Find the **Taylor series** of the function $f(x)=e^x$ at $x=0$. (05 mark)

5.

a. Use the **L'Hospital's rule** to find the following limits.

i. $\lim_{x\to 1}\frac{x^3+x^2-x-1}{x^2+2x-3}$ (04 mark)

ii. $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ (05 mark)

b. Discuss the *differentiability* of the function,

$$f(x)=\begin{cases}2x-3, & x\le 2\\ x^2-3, & x>2\end{cases} \text{ at } x=2.$$ (05 mark)

c. State the mean value theorem. (03 mark)

d. Verify the conditions of **mean value theorem** are satisfied by the function $f(x) = x^2 - 2x + 3$ in $[-2, 2]$; and find a value for $c$ that satisfies the conclusion of the theorem. (05 mark)

e. State the **Rolle's theorem**. (03 mark)

f. Verify the conditions of **Rolle's theorem** are satisfied by the function:

$$f(x) = x^4 - 2x^2, \quad x \in [-2, 2]$$

and determine value of $c$ in $(-2, 2)$ for which $f'(c) = 0$. (05 mark)

6.

a. Which of the following sentences are *propositions*. If a given statement is a proposition, then write down its *truth value*.

i. A cow is an animal. (03 mark)
ii. $5 \in \{1, 2, 4\}$ (03 mark)
iii. May God bless you. (03 mark)

b. What is the *negation* of each of the following propositions?

i. ISIS is a terrorist organization. (02 mark)
ii. $5 + 1 = 6$ (02 mark)
iii. $2 > 7$ (02 mark)

c. Consider the following statements;

$P$: 5 is a prime number
$Q$: 125 is a square number

Find the truth value of the following propositions based on above information.

i. $\sim P$ (02 mark)
ii. $P \wedge Q$ (02 mark)
iii. $P \vee Q$ (02 mark)
iv. $P \Rightarrow Q$ (02 mark)
v. $P \Leftrightarrow Q$ (02 mark)

d. Construct the *truth table* for the statement $(P \vee \sim Q) \wedge P$. (10 mark)